

# A LITTLEWOOD-PALEY TYPE THEOREM FOR BERGMAN SPACES

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ABSTRACT. In this paper, we prove that the original Littlewood-Paley  $g$ -functions can be used to characterize Bergman spaces as well.

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the unit disk in the complex plane  $\mathbb{C}$  with  $\mathbb{T} := \partial\mathbb{D}$  being the unit circle. Recall that for  $0 < p < \infty$ , the Hardy space  $\mathcal{H}^p$  on  $\mathbb{D}$  is defined as the set of all analytic functions  $f$  on  $\mathbb{D}$  satisfying

$$\|f\|_{\mathcal{H}^p} := \sup_{0 \leq r < 1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} < \infty.$$

It is classical that for any  $f \in \mathcal{H}^p$ , almost everywhere on  $\mathbb{T}$  there exist radial limits  $\lim_{r \rightarrow 1^-} f(re^{i\theta})$ , denoted by  $f(e^{i\theta})$ , and there holds the relation  $\|f\|_{\mathcal{H}^p} = \|f\|_{L^p(\mathbb{T})}$ . We refer to [9] for theory of classical Hardy spaces.

Suppose  $f \in \mathcal{H}^p$  and  $f = \sum_n a_n z^n$  is the power series of  $f$ . Consider the following two quantities

$$(1.1) \quad d(f)(z) = \left( \sum_{n=0}^{\infty} |\Delta_n(f)(z)|^2 \right)^{\frac{1}{2}}$$

where  $\Delta_0(f)(z) = a_0$  and  $\Delta_n(f)(z) = \sum_{2^{n-1} \leq k < 2^n} a_k z^k$  for  $n \geq 1$ , and

$$(1.2) \quad g(f)(z) = \left( \int_0^1 (1-r^2) |f'(rz)|^2 dr \right)^{\frac{1}{2}},$$

for  $z \in \mathbb{D} \cup \mathbb{T}$ . Two results on Hardy spaces, essentially due to Littlewood and Paley [6], assert that

$$(1.3) \quad \|f\|_{L^p(\mathbb{T})} \approx \|d(f)\|_{L^p(\mathbb{T})} \quad \text{and} \quad \|f - f(0)\|_{L^p(\mathbb{T})} \approx \|g(f)\|_{L^p(\mathbb{T})},$$

the constants involved being only dependent on  $p$  with  $0 < p < \infty$ . The two equivalent relations (1.3) can be considered to be the beginning of the Littlewood-Paley theory.

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The main purpose of this paper is to prove that these two equivalent relations (1.3) hold true as well in the case of  $L^p(\mathbb{D})$  replacing  $L^p(\mathbb{T})$ , characterizing the so-called Bergman spaces. Recall that for  $0 < p < \infty$ , the Bergman space  $\mathcal{A}^p$  consists of functions  $f$  analytic in  $\mathbb{D}$  with

$$\|f\|_{\mathcal{A}^p} = \left( \int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty$$

where  $dA(z) = dxdy/\pi$  with  $z = x + iy$  in  $\mathbb{D}$ . Note that for  $1 \leq p < \infty$ ,  $\mathcal{A}^p$  is a Banach space under the norm  $\|f\|_{\mathcal{A}^p}$ . If  $0 < p < 1$ , the space  $\mathcal{A}^p$  is a quasi-Banach space with  $p$ -norm  $\|f\|_{\mathcal{A}^p}^p$ .

*Notion.* For two nonnegative (possibly infinite) quantities  $X$  and  $Y$ , by  $X \lesssim Y$  we mean that there exists a positive constant  $C$  depending only on  $p$  such that  $X \leq CY$ , and by  $X \approx Y$  that  $X \lesssim Y$  and  $Y \lesssim X$ .

## 2. MAIN RESULTS

We state our main results as Theorems 2.1 and 2.2.

**Theorem 2.1.** *Let  $0 < p < \infty$ . There are two constants  $A_p$  and  $B_p$  depending only on  $p$  such that*

$$(2.1) \quad A_p \|f\|_{\mathcal{A}^p} \leq \|d(f)\|_{L^p(\mathbb{D})} \leq B_p \|f\|_{\mathcal{A}^p}$$

for any  $f \in \mathcal{A}^p$ .

This characterization of those functions in  $\mathcal{A}^p$  is a straightforward consequence of the first equivalent relation in (1.3), but one of the important features of this characterization is that linear operators obtained by multipliers  $m_k$  (of the coefficients  $a_k$ ) that vary boundedly on the dyadic blocks  $\triangle_n$  preserve the class  $\mathcal{A}^p$ . More generally, this yields a Marcinkiewicz multiplier theorem for Bergman spaces stating that, for any  $0 < p < \infty$  there exists a constant  $C_p$  depending only on  $p$  such that

$$\left\| \sum_{k=0}^{\infty} m_k a_k z^k \right\|_{\mathcal{A}^p} \leq C_p \left( \sup_k |m_k| + \sup_{n \geq 0} \sum_{2^n \leq k < 2^{n+1}} |m_{k+1} - m_k| \right) \|f\|_{\mathcal{A}^p}.$$

The proof of this inequality can be obtained as in the case of Hardy spaces (see for example [9], Theorem XV.4.14).

*Proof of Theorem 2.1.* Let  $0 < p < \infty$ . Denote by  $f_r(z) = f(rz)$  for  $0 < r < 1$  and  $z \in \mathbb{D}$ . By the first equivalent relation in (1.3), one has

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^p dv(z) &= 2 \int_0^1 r dr \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} \\ &\approx 2 \int_0^1 r dr \int_0^{2\pi} |d(f_r)(e^{i\theta})|^p \frac{d\theta}{2\pi} \\ &= \|d(f)\|_{L^p(\mathbb{D})}^p. \end{aligned}$$

This completes the proof of (2.1). □

**Theorem 2.2.** *Let  $0 < p < \infty$ . There are two constants  $\alpha_p$  and  $\beta_p$  depending only on  $p$  such that*

$$(2.2) \quad \alpha_p \|f\|_{\mathcal{A}^p} \leq \|g(f)\|_{L^p(\mathbb{D})} \leq \beta_p \|f\|_{\mathcal{A}^p}$$

for any  $f \in \mathcal{A}^p$  with  $f(0) = 0$ . Consequently,

$$(2.3) \quad \|f\|_{\mathcal{A}^p} \approx |f(0)| + \|g(f)\|_{L^p(\mathbb{D})} \quad \text{for } 1 \leq p < \infty,$$

and

$$(2.4) \quad \|f\|_{\mathcal{A}^p}^p \approx |f(0)|^p + \|g(f)\|_{L^p(\mathbb{D})}^p \quad \text{for } 0 < p < 1.$$

We will deduce this theorem from some classical results, essentially due to Littlewood and Paley, Marcinkiewicz and Zygmund, and a theorem of Coifman and Rochberg [3] on atomic decomposition for Bergman spaces (see Lemma 2.1 below). The proof is thus considerably elementary.

**Lemma 2.1.** *(cf. [7], Theorem 8.3.1) Let  $0 < p \leq 1$ . There exists a sequence  $\{a_k\}$  in  $\mathbb{D}$  and a constant  $C$  such that  $\mathcal{A}^p$  consists exactly of functions of the form*

$$(2.5) \quad f(z) = \sum_{k=1}^{\infty} c_k \frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^{2/p+1}}, \quad z \in \mathbb{D},$$

where  $\{c_k\}$  belongs to the sequence space  $\ell^p$  and the series converges in the quasi-norm topology of  $\mathcal{A}^p$ , and

$$C^{-1} \left( \sum_k |c_k|^p \right)^{\frac{1}{p}} \leq \|f\|_{\mathcal{A}^p} \leq C \left( \sum_k |c_k|^p \right)^{\frac{1}{p}}.$$

*Proof of Theorem 2.2.* We begin with the first inequality in (2.2). Denote by  $f_r(z) = f(rz)$  for  $0 < r < 1$  and  $z \in \mathbb{D}$ . Then by the second equivalent relation in (1.3) we have for any  $0 < p < \infty$ ,

$$\begin{aligned} \int_{\mathbb{D}} |f(z) - f(0)|^p dv(z) &= 2 \int_0^1 \|f_r - f_r(0)\|_{\mathcal{H}^p}^p r dr \\ &\approx \int_0^1 r dr \int_0^{2\pi} \left( \int_0^1 (1 - s^2) |f'_r(se^{i\theta})|^2 ds \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \\ &\lesssim \int_0^1 r dr \int_0^{2\pi} \left( \int_0^1 (1 - s^2) |f'(rse^{i\theta})|^2 ds \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \\ &\approx \|g(f)\|_{L^p(\mathbb{D})}^p. \end{aligned}$$

This proves the first inequality in (2.2).

To prove the second inequality in (2.2) for the case  $0 < p \leq 1$ , we will adopt Lemma 2.1. To this end, we write

$$f_k(z) = \frac{1 - |a_k|^2}{(1 - z\bar{a}_k)^{2/p+1}}.$$

An immediate computation yields that

$$|f'_k(rz)|^2 = (2/p + 1)^2 |\bar{a}_k|^2 (1 - |a_k|^2)^2 \frac{1}{|1 - rz\bar{a}_k|^{2(2/p+2)}}$$

Also, it is easy to check that

$$|1 - tz| \leq (1 - t) + |1 - z| \leq 3|1 - tz|, \quad 0 < t \leq 1, \quad \forall z \in \mathbb{D}.$$

Then

$$\begin{aligned} g(f_k)(z) &= |a_k|(2/p + 1)(1 - |a_k|^2) \left( \int_0^1 \frac{(1 - r^2)dr}{|1 - rz\bar{a}_k|^{2(2/p+2)}} \right)^{\frac{1}{2}} \\ &\lesssim (1 - |a_k|^2) \left( \int_0^1 \frac{dr}{[(1 - r) + |1 - z\bar{a}_k|]^{2(2/p+1)+1}} \right)^{\frac{1}{2}} \\ &\lesssim (1 - |a_k|^2) \frac{1}{|1 - z\bar{a}_k|^{2/p+1}}. \end{aligned}$$

Hence, for  $f = \sum_k c_k f_k$  with  $\sum_k |c_k|^p < \infty$  we have

$$\begin{aligned} \int_{\mathbb{D}} |g(f)(z)|^p dv(z) &\leq \sum_{k=1}^{\infty} |c_k|^p \int_{\mathbb{D}} |g(f_k)(z)|^p dv(z) \\ &\lesssim \sum_{k=1}^{\infty} |c_k|^p (1 - |a_k|^2)^p \int_{\mathbb{D}} \frac{1}{|1 - z\bar{a}_k|^{2+p}} dv(z) \\ &\lesssim \sum_{k=1}^{\infty} |c_k|^p, \end{aligned}$$

where the last inequality is obtained by the fact that

$$\int_{\mathbb{D}} \frac{1}{|1 - z\bar{w}|^{2+p}} dv(z) \approx \frac{1}{(1 - |w|^2)^p} \quad \text{as } |w| \rightarrow 1^-,$$

for  $p > 0$  (see Theorem 1.7 in [5]). By Lemma 2.1, we conclude the second inequality in (2.2) for the case  $0 < p \leq 1$ .

Finally, let  $1 < p < \infty$ . If  $f \in \mathcal{A}^p$ , then  $f$  has the integral representation

$$f(z) = \int_{\mathbb{D}} \frac{f(w)dv(w)}{(1 - z\bar{w})^2}, \quad \forall z \in \mathbb{D}.$$

An immediate computation yields that

$$|f'(rz)| \lesssim (1 - |rz|^2)^{-\frac{1}{2}} \int_{\mathbb{D}} \frac{|f(w)|dv(w)}{|1 - rz\bar{w}|^{\frac{5}{2}}}.$$

Then,

$$\begin{aligned}
g(f)^2(z) &\lesssim \int_0^1 \frac{1-r^2}{1-|rz|^2} \left| \int_{\mathbb{D}} \frac{|f(w)|dv(w)}{|1-rz\bar{w}|^{\frac{5}{2}}} \right|^2 dr \\
&\lesssim \int_{\mathbb{D} \times \mathbb{D}} |f(w)f(\xi)|dv(w)dv(\xi) \int_0^1 \frac{dr}{|1-rz\bar{w}|^{\frac{5}{2}}|1-rz\bar{\xi}|^{\frac{5}{2}}} \\
&\lesssim \int_{\mathbb{D} \times \mathbb{D}} |f(w)f(\xi)|dv(w)dv(\xi) \\
&\quad \times \int_0^1 \frac{dr}{[|1-z\bar{w}|+(1-r)]^{\frac{5}{2}}[|1-z\bar{\xi}|+(1-r)]^{\frac{5}{2}}} \\
&\lesssim \int_{\mathbb{D} \times \mathbb{D}} |f(w)f(\xi)|dv(w)dv(\xi) \\
&\quad \times \left( \int_0^1 \frac{dr}{[|1-z\bar{w}|+(1-r)]^5} \int_0^1 \frac{dr}{[|1-z\bar{\xi}|+(1-r)]^5} \right)^{\frac{1}{2}} \\
&\lesssim \int_{\mathbb{D} \times \mathbb{D}} \frac{|f(w)f(\xi)|}{|1-z\bar{w}|^2|1-z\bar{\xi}|^2} dv(w)dv(\xi) \\
&= \left( \int_{\mathbb{D}} \frac{|f(w)|}{|1-z\bar{w}|^2} dv(w) \right)^2.
\end{aligned}$$

However, the mapping

$$f \mapsto \int_{\mathbb{D}} \frac{f(w)}{|1-z\bar{w}|^2} dv(w)$$

is bounded on  $L^p(\mathbb{D})$  for  $1 < p < \infty$  (e.g. Theorem 1.9 in [5]). Therefore, we conclude the second inequality in (2.2) for the case  $1 < p < \infty$ .  $\square$

- Remark 2.1.** (1) *Since 1930's the Littlewood-Paley theory was developed considerably and mainly carried out by E. M. Stein [8], widening its applicability both in the classical setting involving  $\mathbb{R}^n$  (even when  $n = 1$ ) and in abstract situations involving, among other things, Lie groups, symmetric spaces, diffusion semigroups and martingales. We consult [4] and references therein for more recent information.*
- (2) *Some real-variable characterizations of Bergman spaces involving maximal and area integral functions in terms of the Bergman metric, have been obtained recently by the present authors [1, 2].*

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